Clifford statistics and the temperature limit in the theory of fractional quantum Hall effect

Andrei A. Galiautdinov
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332-0430

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Abstract

Using the recently discovered Clifford statistics we propose a simple model for the grand canonical ensemble of the carriers in the theory of fractional quantum Hall effect. The model leads to a temperature limit associated with the permutational degrees of freedom of such an ensemble. We also relate Schur's theory of projective representations of the permutation groups to physics, and remark on possible extensions of the second quantization procedure.

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In a series of papers, building on the work on nonabelions of Read and Moore [1, 2], Nayak and Wilczek [3, 4, 5] (see also [6] on how spinors can describe aggregates) proposed a startling new spinorial statistics for the fractional quantum Hall effect (FQHE) carriers. The prototypical example is furnished by a so-called Pfaffian mode (occuring at filling fraction $\nu = 1/2$), in which $2n$ quasiholes form an 2^{n-1}-dimensional irreducible multiplet of the corresponding braid group. The new statistics is clearly non-abelian: it represents the permutation group S_N on the N individuals by a non-abelian group of operators in the N-body Hilbert space, a projective representation of S_N.

We have undertaken a systematic study of this statistics elsewhere, aiming primarily at a theory of elementary processes in quantum theory of space-time. We have called the new statistics *Clifford*, to emphasize its intimate relation to Clifford algebras and projective representations of the permutation groups. The reader is referred to [7, 8, 9] for details.

Since the subject is new, many unexpected effects in the systems of particles obeying Clifford statistics may arise in future experiments. One simple effect, which seems especially relevant to the FQHE, might be observed in a grand canonical ensemble of Clifford quasiparticles. In this paper we give its direct derivation first.

Following Read and Moore [2] we postulate that only two quasiparticles at a time can be added to (or removed from) the FQHE ensemble. Thus, we start with an $N = 2n$-quasiparticle effective Hamiltonian whose only relevant to our problem energy level E_{2n} is 2^{n-1}-fold degenerate. The degeneracy of the ground mode with no quasiparticles present is taken to be $g(E_0) = 1$.

Assuming that adding a *pair* of quasiparticles to the composite increases the total energy by ε, and ignoring all the external degrees of freedom, we can tabulate the resulting many-body energy spectrum as follows:

Number of Quasiparticles, $N = 2n$	0	2	4	6	8	10	12	$\cdots$
Degeneracy, $g(E_{2n}) = 2^{n-1}$	0	1	2	4	8	16	32	$\cdots$
Composite Energy, E_{2n}	0ε	1ε	2ε	3ε	4ε	5ε	6ε	$\cdots$

(1)

Notice that the energy levels so defined furnish irreducible multiplets for projective representations of permutation groups in Schur's theory [11], as was first pointed out by Wilczek [4, 5].

We now consider a grand canonical ensemble of Clifford quasiparticles.

The probability that the composite contains n pairs of quasiparticles, is

$$P(n,T) = \frac{g(E_{2n})e^{(n\mu - E_{2n})/k_BT}}{1+\sum_{n=1}^{\infty} g(E_{2n})e^{(n\mu-E_{2n})/k_BT}} \equiv \frac{2^{n-1}e^{-(n\mu-E_{2n})/k_BT}}{1+\sum_{n=1}^{\infty} 2^{n-1}e^{-n(\varepsilon-\mu)/k_BT}}, \tag{2}$$

where μ is the quasiparticle chemical potential. The denominator of this expression is the grand partition function of the composite,

$$Z(T) = 1+\sum_{n=1}^{\infty} g(E_{2n})e^{(n\mu-E_{2n})/k_BT} \equiv 1+\sum_{n=1}^{\infty} 2^{n-1}e^{-n(\varepsilon-\mu)/k_BT} \tag{3}$$

at temperature T.

Now,

$$\sum_{n=1}^{\infty} 2^{n-1}e^{-nx} = e^{-x}\left[2^0e^{-0x}+2^1e^{-1x}+2^2e^{-2x}+\cdots\right] = e^{-x}\ \sum_{n=0}^{\infty} e^{n(\ln 2 - x)}. \tag{4}$$

The partition function can therefore be written as

$$Z(T) = 1+e^{-(\varepsilon-\mu)/k_BT}\sum_{n=0}^{\infty} e^{n(\ln 2-(\varepsilon-\mu)/k_BT)}. \tag{5}$$

This leads to two interesting possibilities (assuming $\varepsilon > \mu$):

1) Regime $0 < T < T_c$, where

$$T_c = \frac{\varepsilon-\mu}{k_B \ln 2}. \tag{6}$$

Here the geometric series converges and

$$Z(T) = \frac{1-e^{-(\varepsilon-\mu)/k_BT}}{1-2e^{-(\varepsilon-\mu)/k_BT}} = \frac{2e^{-(\varepsilon-\mu)/2k_BT}}{1-2e^{-(\varepsilon-\mu)/k_BT}} \sinh\left(\frac{\varepsilon-\mu}{2k_BT}\right). \tag{7}$$

The probability distribution is given by

$$P(n,T) = \frac{2^{n-1}e^{-n(\varepsilon-\mu)/k_BT}(1-2e^{-(\varepsilon-\mu)/k_BT})}{1-e^{-(\varepsilon-\mu)/k_BT}}. \tag{8}$$

2) Regime $T \geq T_c$.

Under this condition the partition function diverges:

$$Z(T) = +\infty, \tag{9}$$

and the probability distribution vanishes:

$$P(n,T) = 0. \tag{10}$$

This result indicates that the temperature T_c of (6) is the upper bound of the intrinsic temperatures that the quasiparticle ensemble can have. Raising the temperature brings the system to higher energy levels which are more and more degenerate, resulting in a heat capacity that diverges at the temperature T_c.

To experimentally observe this effect, a FQHE system should be subjected to a condition where quasiparticles move freely between the specimen and a reservoir, without exciting other degrees of freedom of the system.

A similar limiting temperature phenomenon seems to occur in nature as the Hagedorn limit in particle physics [10].

Knowing the partition function allows us to find various thermodynamic quantities of the quasiparticle system for sub-critical temperatures $0 < T < T_c$. We are particularly interested in the average number of *pairs* in the grand ensemble:

$$\langle n \rangle_{\text{Cliff}} = \lambda \frac{\partial \ln Z}{\partial \lambda}, \tag{11}$$

where $\lambda = e^{\mu/k_B T}$, or after some algebra,

$$\langle n(T) \rangle_{\text{Cliff}} = \frac{e^{-(\varepsilon-\mu)/k_B T}}{(1 - e^{-(\varepsilon-\mu)/k_B T})(1 - 2e^{-(\varepsilon-\mu)/k_B T})}. \tag{12}$$

We can compare this with the familiar Bose-Einstein,

$$\langle n(T) \rangle_{\text{BE}} = \frac{1}{e^{(\varepsilon-\mu)/k_B T} - 1} \equiv \frac{e^{-(\varepsilon-\mu)/k_B T}}{1 - e^{-(\varepsilon-\mu)/k_B T}}, \tag{13}$$

and Fermi-Dirac,

$$\langle n(T) \rangle_{\text{FD}} = \frac{1}{e^{(\varepsilon-\mu)/k_B T} + 1} \equiv \frac{e^{-(\varepsilon-\mu)/k_B T}}{1 + e^{-(\varepsilon-\mu)/k_B T}}, \tag{14}$$

distributions. For the Clifford oscillator, $\langle n(T) \rangle_{\text{Cliff}} \to +\infty$ as $T \to T_c-$, as had to be expected.

Let us now turn to projective representations of the symmetric (permutation) groups that have long been known to mathematicians, but received little attention from physicists. Such representations were overlooked in physics much like projective representations of the rotation groups were overlooked in the early days of quantum mechanics.

For convenience, following [11, 12, 13] (*cf.* also [4, 5]), we briefly recapitulate the main results of Schur's theory.

One especially useful presentation of the symmetric group S_N on N elements is given by

$$\begin{aligned} S_N \quad &= \quad \langle\, t_1, \ldots, t_{N-1} \;:\; t_i^2 = 1,\; (t_j t_{j+1})^3 = 1,\; t_k t_l = t_l t_k \,\rangle, \\ &\quad\quad 1 \le i \le N-1,\; 1 \le j \le N-2,\; k \le l-2. \end{aligned} \tag{15}$$

Here t_i are transpositions,

$$t_1 = (12), t_2 = (23), \ldots, t_{N-1} = (N-1N). \tag{16}$$

Closely related to S_N is the group $\tilde{S}_N$,

$$\begin{aligned} \tilde{S}_N \quad &= \quad \langle\, z, t'_1, \ldots, t'_{N-1} \;:\; z^2 = 1,\; zt'_i = t'_i z,\; {t'_i}^2 = z,\; (t'_j t'_{j+1})^3 = z,\; t'_k t'_l = z\, t'_l t'_k \,\rangle, \\ &\quad\quad 1 \le i \le N-1,\; 1 \le j \le N-2,\; k \le l-2. \end{aligned} \tag{17}$$

A celebrated theorem of Schur (Schur, 1911 [11]) states the following:

(i) The group $\tilde{S}_N$ has order $2(n!)$.

(ii) The subgroup $\{1, z\}$ is central, and is contained in the commutator subgroup of $\tilde{S}_N$, provided $n \ge 4$.

(iii) $\tilde{S}_N / \{1, z\} \simeq S_N$.

(iv) If $N < 4$, then every projective representation of S_N is projectively equivalent to a linear representation.

(v) If $N \geq 4$, then every projective representation of S_N is projectively equivalent to a representation ρ,

$$\begin{aligned}
\rho(S_N) \quad = \quad & \langle\, \rho(t_1), \ldots, \rho(t_{N-1}) : \rho(t_i)^2 = z, (\rho(t_j)\rho(t_{j+1}))^3 = z, \\
& \rho(t_k)\rho(t_l) = z\,\rho(t_l)\rho(t_k)\,\rangle, \\
& 1 \leq i \leq N-1,\ 1 \leq j \leq N-2,\ k \leq l-2,
\end{aligned} \tag{18}$$

where $z = \pm 1$. In the case $z = +1$, ρ is a linear representation of S_N.

The group $\tilde{S}_N$ (17) is called the *representation group* for S_N.

The most elegant way to construct a *projective* representation $\rho(S_N)$ of S_N is by using the complex Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(V, g) \equiv \mathcal{C}_N$ associated with the real vector space $V = N\mathbb{R}$,

$$\{\gamma_i, \gamma_j\} = -2g(\gamma_i, \gamma_j). \tag{19}$$

Here $\{\gamma_i\}_{i=1}^N$ is an orthonormal basis of V with respect to the symmetric bilinear form

$$g(\gamma_i, \gamma_j) = +\delta_{ij}. \tag{20}$$

Clearly, any subspace $\bar{V}$ of $V = N\mathbb{R}$ generates a subalgebra $\mathrm{Cliff}_{\mathbb{C}}(\bar{V}, \bar{g})$, where $\bar{g}$ is the restriction of g to $\bar{V} \times \bar{V}$. A particularly interesting case is realized when $\bar{V}$ is

$$\bar{V} := \left\{ \sum_{k=1}^N \alpha^k \gamma_k \ : \ \sum_{k=1}^N \alpha_k = 0 \right\} \tag{21}$$

of codimension one, with the corresponding subalgebra denoted by $\bar{\mathcal{C}}_{N-1}$ [13].

If we consider a special basis $\{t'_k\}_{k=1}^{N-1} \subset \bar{V}$ (which is *not* orthonormal) defined by

$$t'_k := \frac{1}{\sqrt{2}}(\gamma_k - \gamma_{k+1}), \quad k = 1, \ldots, N-1, \tag{22}$$

then the group generated by this basis is isomorphic to $\tilde{S}_N$. This can be seen by mapping t_i to $t'{}_i$ and z to -1, and by noticing that

1) For $k = 1, \ldots, N-1$:

$${t'_k}^2 = -1; \tag{23}$$

2) For $N-2 \geq j$:

$$(t'_j t'_{j+1})^3 = -1; \tag{24}$$

3) For $N-1 \geq m > k+1$:

$$t'_k t'_m = -t'_m t'_k, \tag{25}$$

as can be checked by direct calculation.

One choice for the matrices is provided by the following construction (Brauer and Weyl, 1935 [14]):

$$\begin{aligned}
\gamma_{2k-1} &= \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes (i\sigma_1) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\
\gamma_{2k} &= \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes (i\sigma_2) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\
& \qquad\qquad\qquad i = 1,\ 2,\ 3, \ldots,\ M,
\end{aligned} \tag{26}$$

for $N = 2M$. Here σ_1, σ_2 occur in the k-th position, the product involves M factors, and σ_1, σ_2, σ_3 are the Pauli matrices.

If $N = 2M+1$, we first add one more matrix,

$$\gamma_{2M+1} = i\,\sigma_3 \otimes \cdots \otimes \sigma_3 \quad (M \text{ factors}), \tag{27}$$

and then define:

$$\begin{aligned}
\Gamma_{2k-1} &:= \gamma_{2k-1} \oplus \gamma_{2k-1}, \\
\Gamma_{2k} &:= \gamma_{2k} \oplus \gamma_{2k}, \\
\Gamma_{2M+1} &:= \gamma_{2M+1} \oplus (-\gamma_{2M+1}).
\end{aligned} \tag{28}$$

The representation $\rho(S_N)$ so constructed is reducible. An irreducible module of $\bar{\mathcal{C}}_{N-1}$ restricts that representation to the irreducible representation of $\tilde{S}_N$, since $\{t'_k\}_{k=1}^{N-1}$ generates $\bar{\mathcal{C}}_{N-1}$ as an algebra [13].

To relate Schur's theory to physics we may try to define a new, purely permutational variable of the Clifford composite, whose spectrum would reproduce the degeneracy of Read and Moore's theory.

A convenient way to define such a variable is by the process of *quantification* (often called second quantization), which is used in all the usual quantum statistics — by mapping the one-body Hilbert space into a many-body operator algebra. This procedure was described in detail in [8].

Let us thus assume that if there is just one quasiparticle in the system, then there is a limit on its localization, so that the quasiparticle can occupy only a finite number of sites in the medium, say $N = 2n$. We further assume that the Hilbert space of the quasiparticle is *real* and $N = 2n$-dimensional, and that a one-body variable (which upon quantification corresponds to the permutational variable of the ensemble) is an antisymmetric generator of an orthogonal transformation of the form

$$G := A \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{bmatrix}, \tag{29}$$

where A is a constant coefficient. Note that in the complex case this operator would be proportional to the imaginary unit i, and the corresponding unitary transformation would be a simple multiplication by a phase factor with no observable effect. Since the quantified operator algebra for $N > 1$ quasiparticles will be complex, the effect of just one such "real" quasiparticle should be regarded as negligible in the grand canonical ensemble.

In the non-interacting case the process of quantification converts G into a many-body operator $\hat{G}$ by the rule

$$\hat{G} \quad := \quad \sum_{l,j}^{N} \hat{e}_l G^l{}_j \hat{e}^j, \tag{30}$$

where usually $\hat{e}_i$ and $\hat{e}^j$ are creators and annihilators, but in more general situations are the generators (that appear in the commutation or anticommutation relations) of the many-body operator algebra. If M_{ij} is the metric (not necessarily positive-definite) on the one-body Hilbert space then

$$\hat{e}_i^\dagger = M_{ij} \hat{e}^j. \tag{31}$$

In the positive-definite case, $M_{ij} = \delta_{ij}$ with $\hat{e}_i^\dagger = \hat{e}^i$, as usual.

In Clifford statistics [8] the generators of the algebra are Clifford units $\gamma_i = 2\,\hat{e}_i = -\gamma_i{}^\dagger$, so it is natural to assume that quantification of G proceeds as follows:

$$\begin{aligned}
\hat{G} &= -A \sum_{k=1}^{n} (\hat{e}_{k+n}\hat{e}^k - \hat{e}_k \hat{e}^{k+n}) \\
&= +A \sum_{k=1}^{n} (\hat{e}_{k+n}\hat{e}_k - \hat{e}_k \hat{e}_{k+n}) \\
&= 2A \sum_{k=1}^{n} \hat{e}_{k+n}\hat{e}_k
\end{aligned}$$

$$\equiv \quad \frac{1}{2} A \sum_{k=1}^{n} \gamma_{k+n} \gamma_k. \tag{32}$$

By Stone's theorem, the generator $\hat{G}$ acting on the spinor space of the complex Clifford composite of $N = 2n$ individuals can be factored into a Hermitian operator $\tilde{O}$ and an imaginary unit i that commutes strongly with $\tilde{O}$:

$$\hat{G} = i\tilde{O}. \tag{33}$$

We suppose that $\tilde{O}$ corresponds to the permutational many-body variable mentioned above, and seek its spectrum.

We note that $\hat{G}$ is a sum of n commuting anti-Hermitian algebraically independent operators $\gamma_{k+n}\gamma_k,\ k = 1, 2, ..., n,\ (\gamma_{k+n}\gamma_k)^\dagger = -\gamma_{k+n}\gamma_k,\ (\gamma_{k+n}\gamma_k)^2 = -1$. If we now use $2^n \times 2^n$ complex matrix representation of Brauer and Weyl (26) for the γ-matrices, we can simultaneously diagonalize the $2^n \times 2^n$ matrices representing the commuting operators $\gamma_{k+n}\gamma_k$, and use their eigenvalues, $\pm i$, to find the spectrum of $\hat{G}$, and consequently of $\tilde{O}$. The final result is obvious: there are 2^{2n} eigenkets of $\tilde{O}$, corresponding to the dimensionality of the spinor space of $\mathrm{Cliff}_{\mathbb{C}}(2n)$. In the irreducible representation of S_N this number reduces to 2^{n-1}, as required by Read and Moore's theory.

Note that in this approach the possible number of the quasiholes in the ensemble is fixed by the number of the available sites, $N = 2n$. A change in that number must be accompanied by a change in the dimensionality of the one-quasiparticle Hilbert space. It is natural to assume that variations in the physical volume of the entire system would privide such a mechanism.

With regard to the quantification procedure (31, 32) mentioned above, we point out that *a priori* there is no compelling reason for using only the formalism of creation and annihilation operators in setting up a many-body theory.

For example, if we choose to work exclusively with an N-body system, then all the initial and final selective actions (projections, or yes-no experiments) on that system can be taken as *simultaneous* sharp production and registration of *all* the N particles in the composite with no need for one-body creators and annihilators. The theory would resemble that of just one particle. The elementary non-relativistic quantum theory of atom provides such an example.

Of course in real experiments much more complicated processes occur. The number of particles in the composite may vary, and if a special *vacuum* mode is introduced, then those processes can conveniently be described by postulating elementary operations of one-body creation and annihilation. Using just the notion of the vacuum mode and a simple rule by which the creation operators act on the many-body modes, it is possible to show (Weinberg [15]) that *any* operator of *such* a many-body theory may be expressed as a sum of products of creation and annihilation operators.

In physics shifts in description are very frequent, especially in the theory of solids. The standard example is the phonon description of collective excitations in crystal lattice. There the fundamental system is an ensemble of a fixed number of ions without any special vacuum mode. An equivalent description is in terms of a variable number of phonons, their creation and annihilation operators, and the vacuum.

It is thus possible that a deeper theory underlying the usual physics might be based on a completely new kind of description. Finkelstein some time ago [16] suggested that the role of atomic processes in such a theory might be played by swaps (or permutations) of quantum space-time events. Elementary particles then would be the excitations of a more fundamental system. The most natural choice for the swaps is provided by the differences of Clifford units (22) defined above.

All that prompted us to generalize from the common statistics to more general statistics, as was done in [8]. There, the quantification rule (31) is a consequence of the so-called representation principle.

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